THE INTERSECTION BETWEEN QUANTIFICATION AND AN ALL-ENCOMPASSING MEANING FOR A GRAPH

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Quantitative reasoning plays a crucial role in students' and teachers' successful modeling activities. In a semester-long teaching experiment with an undergraduate student, we explore how her conception of a graph plays a role in her ability to quantify and maintain quantitative structures. We characterize here Lydia's conception of a graph as one in which the graph entails several quantities she identified in a given dynamic situation, contradicting the conception of a graph as a representation of a multiplicative object consisting of only two quantities. We also discuss her thinking about her graph in terms of figurative and operative thought during a session in which we support her in disembedding and graphically representing quantities.

Keywords: Geometry and Geometrical and Spatial Thinking, Curriculum, Modeling, Teacher Education-Preservice

Introduction

Quantitative reasoning is a crucial component to students and teachers establishing productive meanings (Thompson, 2013). Researchers, however, have found that students' meanings for functions and their graphs lack reasoning about relationships or processes between quantities (Dubinsky & Wilson, 2013; Lobato & Siebert, 2002; Oehrtman, Carlson, & Thompson, 2008; Thompson, 1994b), which ultimately influences students' representational activities. For example, Moore and Thompson (Moore & Thompson, 2015; Thompson, 2016) characterized students' non-quantitative graphing activities in terms of *static shape thinking* (i.e., treating graph-as-wire and focusing on physical features of situations and graphs). During a semester-long teaching experiment, we noted that one of our participants, Lydia, seemed to have a particular meaning for graphs that not only entailed remnants of the static shape thinking discussed by Moore and Thompson, but also included thinking of a graph as containing an abundance of information she perceived in a situation. This latter meaning became problematic as Lydia progressed through the teaching experiment. In this paper, we explore how Lydia's meaning for graphs influenced her reasoning and how quantification and establishment of a graph as a representation of *two* quantities supported her in reasoning quantitatively about the sine and cosine relationship and their graphical representations.

Background and Theoretical Framework

This paper focuses on the intersection of quantification and the consideration of a graph as a multiplicative object. It is important to note that as we define these words, we are operating under the assumption that knowledge is actively constructed in ways idiosyncratic to the knower (von Glasersfeld, 1995). Because of this perspective, we view quantities—conceptions of a specific quality of an object that entails the quality's measurability (Thompson, 1994a)—as personally constructed measurable attributes (Steffe, 1991; Thompson, 2011). Moore and Carlson (2012) highlight the significance of this perspective by arguing that the relationships an individual constructs between quantities depends on her understanding of the quantities and, relatedly, the transformational nature of her image of how these quantities constitute a situation.

Before determining relationships between quantities, one must establish quantities through a process called quantification. Quantification is "the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a

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proportional relationship (linear, bi-linear, multi-linear) with its unit" (Thompson, 2011, p. 37). Consider the Ferris wheel in Figure 1; more specifically, let the object under consideration be the green cart (or more precisely, a point on the Ferris wheel that represents the location of the rider on the wheel). There are many attributes one could observe: colors, shapes, motions, etc. These attributes become a quantity when they are measurable; that is, a quantity is understood as a magnitude or amount-ness, such that it entails a unit or dimension and a way in which to assign numerical value to the magnitude or amount-ness. Note that the process of measuring does not need to be carried out in order for the attribute to be considered a quantity. Some quantities in the Ferris wheel situation include the distance the green cart is above the centerline, the arc length the rider is from the 3 o'clock position on the Ferris wheel, and the speed the rider has traveled. Reasoning about relationships between quantities is termed quantitative reasoning.

A graph is a way for students to represent the relationships between quantities they perceive in a situation. More specifically, a normative Cartesian graph defines a *pair* of quantities—directed lengths—via axes, and each point on the graph is a uniting of two quantity's magnitudes. A cognitive uniting of the two quantities in a given situation is necessary either to construct or to interpret a point on a graph in the aforementioned way. This cognitive uniting of magnitudes is what Saldanha and Thompson (1998) referred to as constructing a multiplicative object. This notion of a multiplicative object stems from Piaget's notion of "and" and as of a multiplicative operator (Piaget & Inhelder, 1963). For instance, the sine relationship can be considered as the cognitive uniting of the vertical distance above the horizontal diameter and the arc length traveled around a circle (both measured relative to the radius of that circle).

A conception of a graph as a multiplicative object along with a robust quantification process is necessary for thinking of graphs operatively (Moore, 2016). Piaget (2001) distinguished between two types of thought, figurative and operative thought. He characterized the former as thought constrained to sensorimotor experiences and perception and the latter as one that prioritizes the coordination of mental operations over figurative activity. For example, conceiving the sine graph as a multiplicative object is an example of operative thought due to the conception entailing the coordination of mental actions in the form of quantitative operations. Static shape thinking is an example of figurative thought, as such thinking is dominated by elements of sensorimotor experience and perception to the extent it does not necessarily entail a relation to Cartesian axes (Moore & Thompson, 2015).

Methods

The results of this study come from a teaching experiment (Steffe & Thompson, 2000), in which we worked with three students (two female, one male) across 10-11 videotaped teaching sessions lasting 1-2 hours. The sessions occurred over the course of a spring semester at a large public university in the southeastern U.S. We conducted two sessions with all three students present. All other sessions included one student with at least two research team members. The students were in their first semester of a four-semester secondary mathematics education program, enrolled in both a content course and a pedagogy course. The students had all completed at least two additional courses beyond a traditional calculus sequence with at least a C as their final grade. We selected students from their first content course based on the research group's analysis of their results on an adapted 1-hour version of Thompson's *Project Aspire* assessment, *Mathematical Meanings for Teaching Secondary Mathematics (MMTsm)* (Thompson, 2016), which focused primarily on questions related to rate of change, interpretation of graphs, symbolic notation, and proportion. The research group analyzed the assessments and identified three students who, from the researchers' perspectives, provided a range of responses and communicated their thinking clearly in their written responses. The three students then agreed to participate in the teaching experiment and were monetarily compensated for their time. The principal investigator of the project was the teacher-researcher at every interview. At least one other member of the research team served as the observer(s). The teacher-researcher and observers (heretofore referred to as researchers) took field notes and asked probing questions as necessary. All sessions and written work were videotaped and digitized.

The goal of the teaching experiment was to create models of individual's mathematics, specifically with regards to students' construction of graphs. Steffe and Thompson (2000) referred to these researcher models as the *mathematics of students* (cf. *students' mathematics*). In both ongoing and retrospective analyses efforts, we analyzed the students' actions using generative and axial approaches (Corbin & Strauss, 2008) in combination with conceptual analysis (Thompson, 2008). We first analyzed the students' observable and audible behaviors in order to develop tentative hypothetical models of their thinking. Then, we attempted to identify connections and consistencies across each student's activities with specific attention to her or his meanings for graphing and the extent to which these meaning entailed quantitative or covariational reasoning. Lastly, we made comparisons across students in order to construct more fine-grained models of the students' thinking. In this report, we focus on one student in the teaching experiment, Lydia, whose meaning for graphs enabled us to explore the intersection between the quantification process and her representational activity as we strived to support a conception of a point on a graph as a multiplicative object.

Results

We divide the results section into three parts: (a) Lydia's initial response to a graphing activity given a dynamic situation from her first interview, (b) a summary of our attempts to support quantitative and covariational reasoning of the sine and cosine relationships through reasoning about the relevant magnitudes in a simplified version of the Ferris wheel situation, and (c) Lydia's attempt to relate what she understood as the sine and cosine relationships in the situation and what she understood as the graph that represented those relationships.

The Ferris Wheel Task

One of the first tasks we presented Lydia was the Ferris Wheel Task, which includes a dynamic image of a Ferris wheel rider (green bucket) who travels at a constant speed clockwise from the bottom of the Ferris wheel (Figure 1) (Desmos, 2014). We first asked her to comment on what she observed in the situation, to which she stated there is "a function that would give us the shape of a circle." We subsequently gave her the prompt: "Graph the relationship between the total distance the rider has traveled *around* the Ferris wheel and the rider's distance from the ground." She then produced what she called her graph in Figure 2. When prompted, she indicated several different total distance and height pairs by pointing on a location on her drawn circle, tracing around her drawn circle from the bottom to indicate total distance, and motioning from the point to the denoted ground to indicate height. She also noted how the speed of the Ferris wheel would influence where along the circle she would be at a particular time, explaining "there has to be some physics formula for that, but I don't know it." Importantly, we inferred her drawn graph and comments to suggest that she conceived one particular curve to describe the Ferris wheel situation as a whole, and from that curve, she could isolate and discuss the quantities under question. As we describe below, this inference is important relative to her response to our prompting her to construct a graph in a normative Cartesian system.

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Figure 1. Animation snapshots of the Ferris wheel task.

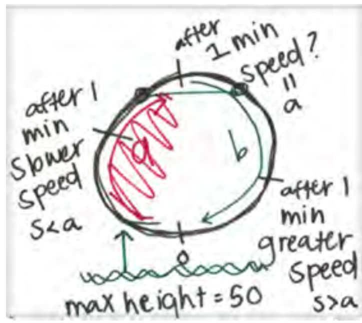

Figure 2. Lydia's initial graph for the Ferris wheel task.

Partitioning Activities with Diagrams of Circles

The teaching experiment tasks shifted away from constructing graphs shortly after this task in an effort to provide Lydia and the other participants of the teaching experiment with the opportunity to focus on reasoning together about the relationships between magnitudes in circular contexts. With considerable support from the researchers, the students engaged in partitioning activities (e.g., Figure 3) with a diagram of the circle to reason about amounts of change in horizontal/vertical distance from the vertical/horizontal diameter for equal changes in arc length. Lydia and another student related these quantities to sine and cosine graphs at the conclusion of the joint sessions. Lydia expressed the novelty of the partitioning activity to her, and it subsequently became a way for her to operate on images (diagrams or graphs) to explore relationships. However, as we argue in the following section, her conception of graphs described in the previous section constrained her ability to use this partitioning activity effectively to describe relationships between quantities.

Figure 3. Partitioning activities using a diagram of a circle from the joint sessions.

Sine Graph

In the next individual session, one week later, the researchers asked Lydia to return to the context of circular motion. Initially, they supported Lydia in drawing a diagram of a situation in which a point on a circle is traveling counterclockwise from the 3 o'clock to the 12 o'clock position (Figure 4a, top). She used her newly developed partitioning activities and constructed changes in horizontal distance for equal changes in arc length, and she compared these changes to conclude that the horizontal distance decreased by an increasing magnitude for an equal change in arc length as the point rotated from the 3 o'clock to the 12 o'clock positions (Figure 4a, top). Shortly after this description, we asked her to create a graph representing the relationship between the horizontal length and the arc length, with our intention being that she produce a normative graph for the cosine relationship. She produced the graph in Figure 4a (bottom), stating that the graph can be comparing "the y-height here [vertical segments in Figure 4a, bottom] and then also can be comparing the x-

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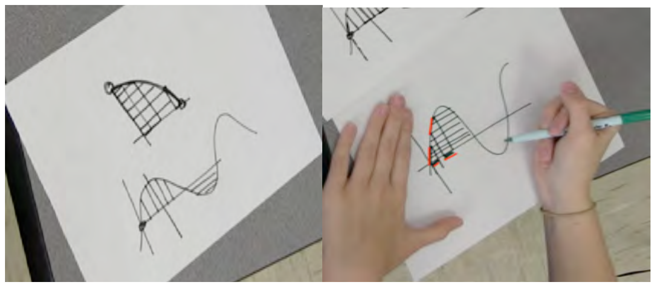
distance [horizontal segments in Figure 4a, bottom].” In her initial description of her drawn graph, Lydia did not reference arc length. Thus, she seemed to indicate that the quantities she had conceived as “*y*-height and *x*-distance” in her diagram of the situation were both represented in her graph. This conclusion is consistent with her conception of her graph from the Ferris Wheel task in which she identified several quantities she thought were represented in her graph.



(a) (b)

Figure 4. (a) (top) Diagram of the situation highlighting amounts of change in “*x*-distance” and (bottom) resulting graph; (b) Equal changes in arc length denoted along curve and corresponding changes marked along horizontal axis.

The researchers subsequently asked Lydia to clarify how she was interpreting “*y*-height and *x*-distance” relative to her drawn graph. She first stated she would show the “*x*-distance” and then traced *along her curve* to indicate “increase in arc length.” Following this statement, she drew in horizontal line segments between the curve, starting nearest to the horizontal axis, and moving upwards (Figure 3b). She described these segments as decreasing as the “arc length” increased, which she argued was the same conclusion she had reached in the circular context. She made this statement while drawing in the vertical lines and motioning along two “arc length” and “*x*-distance” pairs, seemingly mimicking partitioning activities from a previous session (Figure 3). However, when asked to say more about how she was denoting “arc length” and “x-distance” on her graph, specifically about the vertical lines from and highlighted regions in Figure 4b, she questioned the efficacy of her actions. Soon afterwards, she switched from talking about horizontal distances to talking about height, stating, “[S]o we’re doing the arc length and height again [*labeling her axes with arc length on the horizontal axis and height on the vertical axis*]”. She then motioned along her horizontal axis, saying “and as I am going across my arc length” and shortly afterwards tracing along the curve starting from the first maximum in Figure 4b saying, “[O]ur arc length, as it increases, the height will decrease.” She then related this statement to her diagram by completing the first half of the full rotation on her diagram in Figure 4a (top).

To summarize, Lydia stated a directional covariational relationship between (i) “*x*-distance” and “arc length” and (ii) “height” and “arc length” using the same graph. Also, when referring to “*x*-distance”, she denoted horizontal segments that connected two points along her curve and conceived “arc length” as a distance along the curve. When referring to “height”, she conceived “arc length” as a magnitude along the horizontal axis and conceived height as vertical magnitudes between the curve and horizontal axis. After a researcher subsequently drew attention to Lydia’s reference to “arc length” as both a distance along the curve and a distance along the horizontal axis, she was perturbed and over the course of nearly 30 minutes attempted to rationalize the graph entailing the three quantities she had identified (“height”, “arc length”, and “*x*-distance/width”). About seven minutes into her efforts, she made the following statement:

Lydia: I don't know. I'm confused. That's what's going on. I like see the relationship, and I can explain it to a point, and then I get like – I confuse myself with the amount of information I know about a circle that I was just given to me by a teacher, and then what I've like discovered here [*referring to the teaching sessions*] and like how those – I'm like trying to find a connection, but I'm getting confused.

Her extended perturbation emphasizes the figurative conception she has of her graph, relying on perceptual features consistent with static shape thinking to relate it with her diagram. Due to the persistence of Lydia attempting to identify each of the three quantities in her drawn graph and the perturbations she experienced due to this attempt, the researchers decided to draw Lydia's attention back to the situation in hopes of isolating the three quantities she perceived in the situation. The researcher specifically asked Lydia to denote the three quantities (i.e., triple) she identified for various points along the circle (Figure 5).

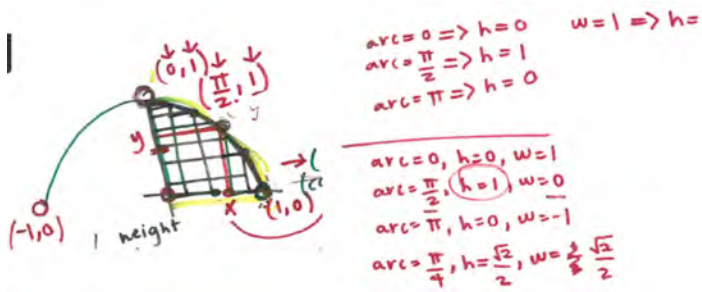


Figure 5. Lydia's diagram of the circle (left) and her triples for various points along the circle (*arc*= arc length, *h*= height, *w*= width).

Upon determining several triples (see Figure 5, bottom right), the researcher asked Lydia how the triples related to her graph (i.e., "[D]oes it represent all three [quantities]? Does it represent just two of them? Does it represent one of them?"). She then drew attention to the origin on her graph and explained:

Lydia: Because this is my – This is *x* – um, *x-y* plane, then here I'm saying at this point [*the origin*], my width is 0, my arc length is 0, and my height is 0.
Researcher: Width is 0, my arc length is 0 and my height is 0.
Lydia: Wait, but then I said at arc length 0, and [*laughs*] height is 0, then my width should be 1.
Researcher: And your width should be 1, right? What about at pi-halves? What should we have?
Lydia: Then I should have a height of 1 [*pointing to curve for an abscissa value of pi/2*].
Researcher: Okay.
Lydia: And then my width should be 0. So this graph does not do anything with the *x-y* plane.
[*Lydia summarizes this claim and then the researcher asks Lydia to consider an arc length of pi radians.*]
Lydia: Then my arc length on the *x*-axis [*motions across horizontal axis*] should be pi. My height should be 1 – or 0, and then my *x*-value should be negative 1. So this [*referring to her drawn graph*] just doesn't have – then this doesn't relate to the *x*, the width [*referring to width from the situation*], just this graph. So my whole circle talks about width and height and arc, but then this graph itself only talks about arc and height. [*speaking emphatically*] Done it. [*laughs*]

We infer that Lydia accommodated her meaning for her drawn graph, including how it related to the circle situation, during this interaction. Specifically, she came to understand that her drawn graph related two particular quantities—arc length and height directed horizontally and vertically, respectively—in a way compatible with the situation. She simultaneously held in mind that these two particular quantities were a subset of the three she understood to constitute the situation. We infer

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that her accommodation occurred when assimilating a point on the circle as containing a trio of information and then interpreting the point on her graph as entailing the three quantities' values as 0 (i.e., the arc along the curve, the abscissa value, and the ordinate value). This resulted in her experiencing a perturbation with her understanding of the situation (i.e., an arc length of 0, a height of 0, and a width of 1). Alleviating this perturbation required that she disembed two of the quantities from the situation, understand these quantities as represented orthogonally with respect to her drawn graph, and conceive a point on her graph as representing each quantity's value or magnitude simultaneously (i.e., as a multiplicative object). She tested the viability of this new model with two additional points before being confident in the efficacy of her actions; each point on her graph was a uniting of two, and only two, quantities. At this point, her thinking about her graph shifted from figurative to operative.

Discussion

We highlight four important findings from Lydia's activity. Firstly, we note the difficulty a student has in maintaining a consistent quantitative structure within and between a situation and its representation when (i) quantification and quantitative reasoning is a novel way of thinking and (ii) one has a conception of a graph as encompassing an abundance of quantities in a situation. For instance, Lydia had a graph on which she attempted to quantify based on the results of her partitioning activities in her diagram of the situation, but she did not maintain the quantities she believed the graph represented as evidenced by her switching arc length from axes to along the curve. Relatedly, the process of disembedding quantities was crucial for Lydia to view her graph as a representation of a multiplicative object, which was what shifted her thinking from figurative to operative. Thirdly, we have provided a more detailed example of Moore and Carlson (2012), who primarily characterized how the quantitative structure of the situation that the student constructed influenced their mathematical artifacts, including how the students conceived of a quantitative invariance between the two. We extend that work by providing a more detailed look into the students' activity by drawing figurative and operative distinctions, thus not presuming the students to have constructed and maintained quantities. Lastly, this study has important implications for the study of trigonometry in that students should understand the cosine (and sine) relationships as disembedding from the unit circle. Lydia's case shows that this disembedding process should not be taken as a given.

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